Consensus protocols in networks with 1-to-n joint-agent interactions*

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***Abstract*—In the scientific literature of the past two decades many conditions to assess consensus-ability in multi-agent networks with 1-to-1 interactions have been proposed. More recently the framework has been extended to include the m-to-1 type-interactions, referred as joint-agent interactions. In this note we consider a novel framework of networks with 1-to-n interactions, potentially extendable to m-to-n ones, and formulate sufficient and necessary analytical conditions for consensus-ability. The considered interactions may also well represent the heterophilous dynamics in opinion and social systems. Finally, shedding light on the scenario of 1-to-n interactions could inspire the formulation of new distributed algorithms to be exploited in engineering applications.**

I. Introduction and paper contribution

Consensus protocols [1] have received a great deal of attention in recent years and find application in many areas of science and engineering, ranging from multi agent systems coordination [2], distributed optimization [4], [5], smart grid and sensor networks [6], [7], social behavior analysis by examining the interplay between strategic and social imitative actions driving the system towards consensus [8], to opinion formation in social networks [9], [10] (just to cite a few). Broadly speaking, these are mechanisms of interaction within a multitude of agents which prescribe how the value of a private local variable "owned" by individual agents is updated in response to interaction with its neighbours (in particular in response to communications by neighbouring agents of their own private variable values). If such interactions are suitably designed it is possible to achieve so called consensus, namely the values of all agents' variables will converge to a single value in common to all agents. To make this model precise and amenable to quantifiable updates and predictions, interaction strength are formulated mathematically as functions of agents values, and research in consensus has focused on identifying under what 'shape' of agents' interaction and more significantly which kind of agents' interaction topologies enable the ability of a network to always reach consensus, no matter what the initial distribution of initial agents' values is considered.
Classically only 1-to-1 agent interactions are considered. Namely, the strength of interaction only depends on the (relative) values of two nodes, the influencer and the influenced. For instance, if i is the influencer and j is the influenced agent, we might denote with x_i and x_j the values of the corresponding variables. Hence, the strength of interaction could be proportional to $x_i - x_j$, or more in general be computed through some nonlinear function $g(x_i - x_j)$ of their relative values. More in general $g(x_i, x_j)$ could also be allowed. We denote such 1-to-1 interaction as $i \rightarrow j$. It has been observed that the formation of social groups is facilitated by the interaction among individuals with dissimilar attributes, resulting in the heterophily principle [14], [15]. Such behaviour is well represented by the diffusive 1-to-1 interaction proportional to $x_i - x_j$, which indeed occur among agents with sufficiently dissimilar attributes or opinions x_i (i.e. heterophilic interaction [11]). In this regard, several works in the literature, with both speculative and analytical outcomes, have been devised for networks with one-to-one heterophilous interactions [12], [13]. In the engineering context of parallel distributed computing and in the social one of opinion and complex contagion dynamics, it makes sense to consider more general kind of interactions. This can be beneficial, for instance, in order to robustify the consensus-ability of the network [17], [18]. Recently, the notion of joint-agent interactions was proposed, in particular, the idea that a group of agents might be influencing a single agent. These type of interactions, also called m-to-1 interactions, give rise to an influence from the group towards the agent whenever all agents in the group have values which are greater than that of the influenced agent, or symmetrically when all agents in the group have values which are smaller than that of the influenced agent [19], [20]. From the mathematical point of view, this notion can be further generalized to model the interaction of two disjoint groups of agents, I the influencing group and J the group being influenced, $I \rightarrow J$. Let x_i with $i \in I$ denote the variables of the influencers and respectively x_j, $j \in J$ the variables of the influenced group. We denote by:

$$\bar{x}_I := \max_{i \in I} x_i, \qquad \underline{x}_I := \min_{i \in I} x_i$$

and similarly:

$$\bar{x}_J := \max_{j \in J} x_j, \qquad \underline{x}_J := \min_{j \in J} x_j.$$

One can postulate a strength of interaction which is different from 0 as long as:

$$[\underline{x}_I, \bar{x}_I] \cap [\underline{x}_J, \bar{x}_J] = \emptyset,$$

viz. when the convex hulls of the values of the two groups of agents are disjoint. As it turns out, the details of the strength of

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interaction are immaterial but, to fix ideas, one might consider a linearly growing influence of the kind:

$$g_{I\to J}(x) := \max\{\underline{x}_I - \bar{x}_J, 0\} + \min\{\bar{x}_I - \underline{x}_J, 0\}.$$

Ideally one would like to consider for any given finite group of agents, an arbitrary set of m to n interactions among them. The term "m to n" refers to the fact that both influencing agents I and influenced agents J can be sets of cardinality bigger than 1. Even before giving technical details of the update equations for the individual network's variables, this is a very rich class of distributed parallel protocols, with intrinsically nonlinear dynamics whose potential asymptotic behaviour is difficult to assess a priori. The consensus-ability of the network has been well understood in the case of m-to-1 interaction, leading to the formulation of purely topological conditions that characterize when a network can achieve consensus regardless of initial conditions [19]. These conditions are formulated by using the language of Petri Nets (essentially used as bipartite graphs, rather than as a modelling tool for Discrete Event Systems). In this context, the paper contribution may be summarized as follows:

1 to consider, dually with respect the existing literature, the scenario of networks only involving 1-to-n type-interactions that, to the authors best knowledge, is novel in the consensus framework. We formulate a sufficient and a necessary condition for the consensus-ability of such networks;

2 such kind of interactions may well represent and extend the opinion and social complex systems with heterophilous dynamics [11], [16]. In this respect we largely extend the nature (nonlinear) and type (1-to-n) of network heterophilic interactions, formulating sufficient and necessary analytical conditions for consensus-ability;

3 as by-product, shedding light on the scenario of 1-to-n interactions could inspire the formulation of new distributed algorithms to exploit in engineering applications as well as assessed via m-to-1 interactions [17].

The rest of paper is organized as follows. In Sections II and III the background, notation and framework are presented, respectively. In Sections IV and V are formulated the sufficient and necessary conditions in the novel scenario. Finally, conclusions and future works are outlined in Section VI.

II. PETRI NET BACKGROUND AND FORMALISM

In the theory of Discrete Event systems, Petri Nets are token passing devices that model the evolution of the so called "marking" of the Petri Net under the action of asynchronous events (also called firing transitions). In the study of consensus, we merely use their structural features, essentially that of a bipartite graph, and adopt the associated jargon as many of the concepts needed in this analysis have already been developed and formulated in such context. A Petri Net is a quadruple, $\{P, T, E, F\}$, where P is a finite set, $\{p_1, \ldots, p_N\}$, called the 'places', which we identify with the agents of the network, T, is a finite set, $P \cap T = \emptyset$ which we identify with the interactions of the network. The set E fulfills $E \subseteq P \times T$, and $(p, t) \in E$ means that agent p is an influencer in the interaction t; dually, $F \subseteq T \times P$, and $(t, p) \in F$ means that agent p is influenced by interaction t. For a subset $S \subset P$, we denote by S^c the complement of S in P, viz. $P \backslash S$. Similar notation is adopted for a subset of T. We denote by $I(t)$ the set of influencers of interaction t (input places) and by $O(t)$ the set of agents influenced in interaction t (output places). A similar notation is introduced for places, meaning for $p \in P$, $I(p)$ denotes the set of interactions that influence p, while $O(p)$ denotes the set of interactions that see p as an influencing agent. Moreover, $I(\cdot)$ and $O(\cdot)$ notations are extended to sets by denoting for $Q \subseteq P$ or :

$$I(Q) := \bigcup_{p\in Q} I(p) \qquad O(Q) := \bigcup_{p\in Q} O(p),$$

and similarly for $I(S \subseteq T)$ and $O(S \subseteq T)$.

Petri Nets are used throughout this paper because of their expressive power to fully capture the set of 1 (or m)-to-n interactions. They are not adopted as a model of Discrete Event Systems (DES). In fact, an isomorphic alternative formalism could be that of directed hyper-graphs, where places are mapped to vertices and transitions are mapped to hyper-edges. We chose the formalism of Petri Nets as the relevant structural notions involved (namely traps and siphons, [21]) have been studied in depth in the context of DESs.

In the case of n to 1 interaction the following concept is important.

***Definition** 1:* A set $S \subseteq P$ is a *siphon* of the Petri Net if it fulfills:

$$I(S) \subseteq O(S). \tag{1}$$

This means that siphons are sets that only accept infuences provided at least one of the influencers is part of the siphon itself. It turns out that the dual notion of *trap* will play a role in the study of networks with 1-to-n interactions:

***Definition** 2:* A set $S \subseteq P$ is a *trap* of the Petri Net if it fulfills:

$$O(S) \subseteq I(S). \tag{2}$$

In words this means that agents in a trap can only influence groups that contain at least somebody from the trap. Given two traps T_1, T_2 we say that they are *covering* if $T_1 \cup T_2 = P$. We will see that understanding the lay out of pairs of covering traps is crucial to characterize the emergence of asymptotic consensus in the considered set-up.

Notice that $O(P) = T = I(P)$ (namely every interaction comes from some agent and influences at least one agent); hence P is both a trap and a siphon; we refer to it as the trivial trap, or siphon. To illustrate the concept consider the Petri Net shown in Fig. 1. This net has 4 places, viz. $P = \{p_1, p_2, p_3, p_4\}$, and 4 transitions, $T = \{t_1, t_2, t_3, t_4\}$. For each transition t_i we have the edge $(p_i, t_i) \in E$ as well as two outgoing edges in F, $(t_i, p_{i \bmod 4+1})$ and $(t_i, p_{(i+1) \bmod 4+1})$ (where $a \bmod b$ denotes the remainder of the division between positive integers a and b). Accordingly we have the following traps of minimal support $T_1 := \{p_1, p_3\}$, $T_2 := \{p_2, p_4\}$. Additional traps, are all subsets of P which include either T_1 or T_2. There is only one siphon which is P itself.

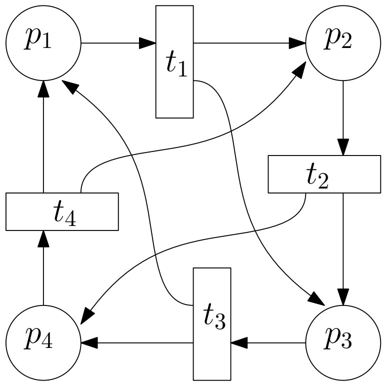

Fig. 1. Example of Petri Net

III. Network's model and consensus definition

Let $P = \{p_1, \ldots, p_N\}$ denote the set of places, which we identify with agents, and x_i, $i \in \{1, \ldots, N\}$ denote the corresponding variables. We continuously update their values in time through a differential equation, which can be specified as follows:

$$\dot{x}_i(t) = \sum_{\tau \in I(p_i)} g_{I(\tau) \to O(\tau)}(x(t)) \qquad i \in \{1, \ldots, N\} \tag{3}$$

Notice that, for each agent p_i, the subset $I(p_i) \subset T$ models the interactions, coming from neighbouring agents, that affect its evolution. In the context of 1-to-1 interactions, multiagent networks are modeled as standard (non-bipartite) graphs, and interactions $i \to j$ are associated with the corresponding edge of the graph. In this generalised set-up, however, a bipartite structure is needed to fully represent influences $i \to J$ where J may have cardinality bigger than 1. In particular, each element of $I(p_i) \subset T$ is associated to an interaction which is affecting node i, viz. $j \to I$ with $i \in I$ (for some agent j). In addition, while an additive aggregation of influences is perhaps the most typical instance of processing to be carried out by individual agents, this is not the only possibility. For instance:

$$\dot{x}_i(t) = \max_{\tau \in I(p_i)} g_{I(\tau) \to O(\tau)}(x(t)) + \min_{\tau \in I(p_i)} g_{I(\tau) \to O(\tau)}(x(t)) \tag{4}$$

$$i \in \{1, \ldots, N\}$$

is also a plausible choice or any other function h_i of $|I(p_i)|$ arguments, with the following properties:

- h_i is monotone non-decreasing with respect to each of its arguments and Lipschitz continuous
- $h_i(0) = 0$;
- h_i is strictly monotone at zero:

$$h_i(0, \ldots, a, \ldots, 0)a > 0,$$

  for all $a \neq 0$, and for all positions of a as an argument of h_i.

Hence, we may write a generalized form of update equations as follows:

$$\dot{x}_i(t) = h_i(\ldots, g_{I(\tau) \to O(\tau)}(x(t)), \ldots) \qquad \tau \in I(p_i). \tag{5}$$

While this affords an extra layer of generality to our results, the techniques are novel and non-trivial even in the simpler case of additive interactions represented by equation (3) and we recommend the readers first familiarize with this set-up before reassessing the more general situation arising from equations such as (4) or (5).

***Remark** 1:* When considering an influence $i \to J$, it is useful to realise that agents which belong to J might not be aware of the group J, or of the states of nodes within the group J. In other words, group J might just be known by agent i, which then will collect neighbouring agents' states, compute the quantity $g_{i \to J}(x(t))$ and broadcast this to each agent in J. Individual agents will then use this quantity to update their states. This, however, entails knowledge on the part of the influencing agent of the state of every influenced agent.

Alternatively, agents in J may be aware of the existence of the group J and of it being susceptible to influence from i and use knowledge of $x_i(t)$ and $x_J(t)$ to locally compute the strength $g_{i \to J}(x(t))$. In this case agent i does not need to know the values of $x_J(t)$ but only needs to broadcast its own state to all influenced agents.

For any initial condition $x_0 \in \mathbb{R}^N$ we denote the solution of (5) which fulfills $x(0) = x_0$ by $\varphi(t, x_0)$, viz. $x(t) = \varphi(t, x_0)$.

***Definition** 3:* We say that the network achieves asymptotic consensus for a given initial condition x_0, whenever there exists $x_\infty \in \mathbb{R}$:

$$\lim_{t \to +\infty} \varphi(t, x_0) = x_\infty \mathbf{1}, \tag{6}$$

where $\mathbf{1}$ denotes the vector of all ones in $\mathbb{R}^N$.

We aim to find conditions on the topology of 1-to-n interactions under which consensus is achieved for all $x_0 \in \mathbb{R}^N$.

The problem of characterising consensus-ability of any network with respect to the topology of interactions considered, can be addressed by considering the following notion.

***Definition** 4: (Covering Traps Compatibility Property)* We say that a network fulfills the covering traps compatibility property if, for any pair of (non-trivial) covering traps $T_1 \neq T_2$ (viz. $T_1 \cup T_2 = P$ and $T_1, T_2 \subsetneq P$) at least one of the following conditions hold:

1) $O(T_2 \backslash T_1) \cap I(T_1 \backslash T_2) \not\subseteq I(T_2 \backslash T_1)$;
2) $O(T_1 \backslash T_2) \cap I(T_2 \backslash T_1) \not\subseteq I(T_1 \backslash T_2)$.

In words this means that for a network with covering traps compatibility whenever two distinct covering traps are considered, at least some influence goes from $T_2 \backslash T_1$ into $T_1 \backslash T_2$ which does not influence nodes in $T_2 \backslash T_1$ or viceversa. Notice that two disjoint traps covering P violate the covering traps compatibility property. In fact, $T_1 \backslash T_2 = T_1$ and $T_2 \backslash T_1 = T_2$. However, since T_1 is a trap we have, $O(T_1) \cap I(T_2) \subseteq I(T_1) \cap I(T_2) \subseteq I(T_1)$, which violates condition 2. Similarly, since T_2 is a trap we see: $O(T_2) \cap I(T_1) \subseteq I(T_2)$, which violates condition 1. Hence the Petri Net in Fig. 1 does not enjoy the Covering Traps Compatibility Property. Instead, the same cyclic topology, with 3 places and 3 transitions, (see Fig. 2) yields the following traps: $\{p_1, p_2\}$, $\{p_2, p_3\}$ and $\{p_1, p_3\}$. Choosing, without loss of generality (due to the

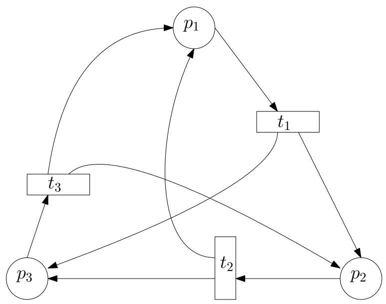

Fig. 2. Cyclic Petri Net with 3 places and 1-to-2 interactions.

cyclic symmetry of the net), $T_1 = \{p_1, p_2\}$ and $T_2 = \{p_2, p_3\}$ we have $T_1 \backslash T_2 = \{p_1\}$, and $T_2 \backslash T_1 = \{p_3\}$. We can then verify that:

$$O(T_1 \backslash T_2) \cap I(T_2 \backslash T_1) = O(p_1) \cap I(p_3) =$$
$$\{t_1\} \cap \{t_1, t_2\} = \{t_1\} \not\subseteq \{t_2, t_3\} = I(T_1 \backslash T_2)$$

A similar verification can be done for all pairs of covering traps. Hence, this Petri Net fulfills the Covering Traps Compatibility.

***Remark** 2:* Traps (and siphons, which are simply obtained from traps by swapping E and F arcs) are structural invariant of Petri Nets. Algorithms for their computation have been proposed in the DES literature, [21], and complexity of their computation is known to grow combinatorially in some cases. While Petri Nets adopted throughout this paper have a single incoming arc entering every transition, to the best of our knowledge there is no specific algorithm that exploits such feature to reduce the complexity of traps computation. Hence one relies on general purpose algorithms for traps computation. Similarly, we do not know of algorithmic ways to directly verify the Covering Traps Compatibility property without explicit computation of individual traps.

We are now ready to state our main result which will be proved in the next Section through a series of intermediate steps.

***Theorem** 1:* Consider the set of differential equations, as defined in equation (5), and the associated Petri Net. Then, a sufficient condition for the solution of (5) to achieve asymptotic consensus irrespective of the initial condition $x_0 \in \mathbb{R}^N$ is that the Covering Traps Compatibility Property holds for the associated Petri Net.

We informally refer to the property of achieving asymptotic consensus from arbitrary initial conditions as consensusability.

***Remark** 3:* It is interesting to interpret the Covering Traps Compatibility Property in the context of networks with 1-to-1 interactions. Under this assumption standard graphs are adequate to represent neighbours influences. Moreover, asymptotic consensus is achieved (regardless of initial conditions) if and only if the associated graph admits a spanning tree.
In particular, whenever all interactions are of 1-to-1 type, traps can be recognized directly from the associated graph and a set of nodes T is a trap (of the Petri Net) if and only if it does not have outgoing arcs in the associated graph (viz. $I(T^c) = \emptyset$). We claim that the Covering Traps Compatibility is equivalent to existence of a spanning-tree in the associated graph of interactions.
To see this, assume a spanning tree exists. Let $r \in \mathcal{P}$ be its root. For any pair of covering traps T_1, T_2, r must belong to at least one of them, say $r \in T_1$, without loss of generality. However, every other node can be reached from r, hence, $T_1 = P$, viz. T_1 is a trivial trap. Hence, any pair of covering traps must include the trivial trap P, and the traps compatibility property is automatically satisfied. Conversely, if the trap compatibility property is fulfilled, then asymptotic consensus is achieved from arbitrary initial conditions, and that, in turn, implies existence of a spanning tree in the corresponding interaction graph.
As a simple example of a network with 1-to-1 interactions not fulfilling the traps compatibility property consider 3 agents $\{1, 2, 3\}$, such that $1 \to 3$ and $2 \to 3$ are the only interactions (1-to-1). The associated Petri Net and graph are shown in Fig. 3. Clearly the associated graph does not have a spanning-tree as two leaders $1, 2$ compete to influence a follower, 3. Analysing through the lens of Petri Nets and traps, we have 2 non trivial traps, $T_1 = \{p_1, p_3\}$ and $T_2 = \{p_2, p_3\}$, shown as dotted lines in the Figure. Notice that $O(T_1 \backslash T_2) \cap I(T_2 \backslash T_1) = O(\{p_1\}) \cap I(\{p_2\}) = \{t_1\} \cap \emptyset = \emptyset \subseteq I(T_1 \backslash T_2)$. Similarly, $O(T_2 \backslash T_1) \cap I(T_1 \backslash T_2) = \emptyset \subseteq I(T_2 \backslash T_1)$. This shows that the covering trap compatibility does not hold.
Dually, consider the network of 1-to-1 interactions $3 \to 1$ and $3 \to 2$, viz. a leader 3, influencing followers 1 and 2. The associated Petri Net and graph are shown in Fig. 3. There are only 2 non-trivial traps, $T_1 = \{p_1\}$ and $T_2 = \{p_2\}$. Together they do not cover the full set of agents, P. Hence, the covering traps compatibility property trivially holds and this is in agreement with the fact that a spanning tree exists in the corresponding graph.

Of course, if only 1-to-1 interactions are present, testing existence of spanning trees is more intuitive and direct than adopting the Petri Net point of view. This discussion is to emphasize that while the Covering Trap Compatibility is a more general tool, capable of dealing with genuinely new nonlinear consensus dynamics, remarkably it boils down to the well-known necessary and sufficient conditions for consensus in the case of standard 1-to-1 interactions. Finally, to validate Theorem 1 we show the simulation of the net in Fig. 2, (see Fig. 4).

IV. Sufficiency proofs and preliminary results

The following Lemma is a basic result that allows to analyze consensus protocols. It holds for arbitrary m to n interaction networks of the type considered.

***Lemma** 1:* Consider an arbitrary Petri Net, and the associated set of differential equations, as defined in equation (5). Then, for any M and m in $\mathbb{R}$ the following sets are forward invariant:

$$X_{\leq M} := \{x : \max_i x_i \leq M\}$$

$$X_{\geq m} := \{x : \min_i x_i \geq m\}.$$

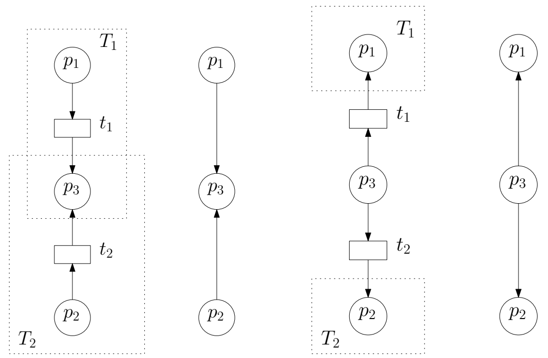

Fig. 3. Petri Nets and graph networks with 1-to-1 interactions violating and fulfilling trap compatibility

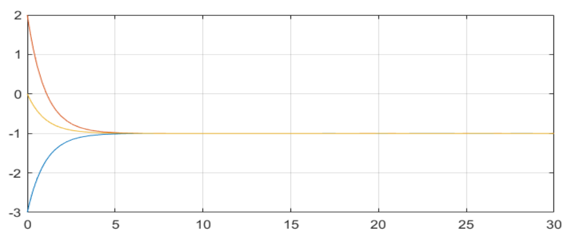

Fig. 4. Simulation of cyclic network in Fig. 2

Take an arbitrary initial condition x_0 in $\mathbb{R}^N$. Denote by $x(t) := \varphi(t, x_0)$. We may define $\bar{x}(t) := \max_{i \in N} x_i(t)$, and $\underline{x}(t) := \min_{i \in N} x_i(t)$. By virtue of Lemma 1, the function $\bar{x}(t)$ is monotonically non-increasing and $\underline{x}(x)$ is monotonically non-decreasing.

The notion of ω-limit set is needed for the subsequent analysis. We recall its definition before employing it in the Lemmas that follow:

$$\omega(x_0) = \{x \in \mathbb{R}^N : \exists \{t_n\}_{n=1}^{+\infty} : t_n \to +\infty \text{ and } \varphi(t_n, x_0) \to x\}.$$

***Lemma** 2:* Consider a set of differential equations, as defined in equation (5), and the associated Petri Net. Consider any initial condition $x_0 \in \mathbb{R}^N$. Then $x(t) := \varphi(t, x_0)$ is uniformly bounded, $\omega(x_0)$ exists non-empty, and the following limits are well defined:

$$\lim_{t \to +\infty} \bar{x}(t) = \bar{x}_\infty,$$
$$\lim_{t \to +\infty} \underline{x}(t) = \underline{x}_\infty.$$

Moreover, for every $x \in \omega(x_0)$ the following sets are non-empty:

$$\bar{\mathcal{I}}(x) := \{p_i \in P : x_i = \bar{x}_\infty\},$$
$$\underline{\mathcal{I}}(x) := \{p_i \in P : x_i = \underline{x}_\infty\}.$$

Proof. By Lemma 1, the following holds for any $i \in N$ and all non-negative t such that solutions are well defined:

$$\bar{x}(0) \geq \bar{x}(t) \geq x_i(t) \geq \underline{x}(t) \geq \underline{x}(0).$$

Hence, solutions are bounded and defined for all $t \geq 0$ (no finite escape time). Moreover, due to monotonicity, $\bar{x}(t)$ and $\underline{x}(t)$ both admit a limit as $t \to +\infty$.

Given $x \in \omega(x_0)$ arbitrary, let $t_n \to +\infty$ be such that $x(t_n) \to x$. We see, by continuity of the max function:

$$\bar{x} = \max_i \left[\lim_{n \to +\infty} x(t_n)\right]_i = \lim_{n \to +\infty} \max_i x_i(t_n)$$
$$= \lim_{n \to +\infty} \bar{x}(t_n) = \bar{x}_\infty.$$

This proves that $\bar{\mathcal{I}}(x)$ is non-empty. A similar argument applies to $\underline{\mathcal{I}}(x)$.

***Lemma** 3:* For any $x \in \omega(x_0)$ the sets $\bar{\mathcal{I}}(\varphi(t, x))$ and $\underline{\mathcal{I}}(\varphi(t, x))$ are monotonically non-increasing with respect to set inclusion as a function of t.

Proof. Consider the case of $\bar{\mathcal{I}}(\varphi(t, x))$. Assume, that element i does not belong to $\bar{\mathcal{I}}(\varphi(0, x))$. In particular, $x_i(0) < \bar{x}_\infty$. For every $x \in \omega(x_0)$ we have $f_i(x + e_i(\bar{x}_\infty - x_i)) \leq 0$ as all agents have values $\leq \bar{x}_\infty$. Hence, for every $t \geq 0$, exploiting invariance of $\omega(x_0)$ and Lipschitzness of f_i,

$$\dot{x}_i(t) \geq [f_i(x(t)) - f_i(x(t) + e_i(\bar{x}_\infty - x_i(t)))] \geq -L|\bar{x}_\infty - x_i(t)|.$$

This shows that $\bar{x}_\infty - x_i(t) \geq e^{-Lt}(\bar{x}_\infty - x_i(0)) > 0$ for all $t \geq 0$ and therefore i does not belong to $\bar{\mathcal{I}}(\varphi(t, x))$ for all $t \geq 0$. By arbitrariety of i, this proves the Lemma.

***Remark** 4:* For any $x \in \omega(x_0)$ and assuming $\underline{x}_\infty = \min_i x_i < \max_i x_i = \bar{x}_\infty$, the following inclusions are of interest:

$$\bar{\mathcal{I}}(x) \subseteq \underline{\mathcal{I}}(x)^c,$$
$$\underline{\mathcal{I}}(x) \subseteq \bar{\mathcal{I}}(x)^c.$$

In other words, agents attaining the minimum are included in the set of agents not attaining the maximum and vice-versa.

The following Lemma is useful to characterize the sets $\bar{\mathcal{I}}(\varphi(t, x))$ and $\underline{\mathcal{I}}(\varphi(t, x))$.

***Lemma** 4:* Let there be a non-empty interval $J \subseteq \mathbb{R}$ and $x \in \omega(x_0)$, such that $\bar{\mathcal{I}}(\varphi(t, x))$ (respectively $\underline{\mathcal{I}}(\varphi(t, x))$) is constant for all $t \in J$. Then $\bar{\mathcal{I}}(\varphi(t, x))^c$ (respectively $\underline{\mathcal{I}}(\varphi(t, x))^c$) is a trap of the associated Petri Net (for $t \in J$).

Proof. By contradiction, if this was not the case there would be one transition $t_j \in T$ such that $t_j \in O(\bar{\mathcal{I}}(\varphi(t, x))^c)$, such that $O(t_j) \cap \bar{\mathcal{I}}(\varphi(t, x))^c = \emptyset$, viz. $O(t_j) \subseteq \bar{\mathcal{I}}(\varphi(t, x))$. Hence, for all agents i in $\bar{\mathcal{I}}(\varphi(t, x))$ we see that:

$$\dot{x}_i(t) \leq h_i(0, \ldots, 0, g_{I(t_j) \to O(t_j)}(x(t)), \ldots, 0) < 0$$

which contradicts the fact that all agents achieving the maximum have constant value for all t. The inequality follows by realising that $I(t_j) \notin \bar{\mathcal{I}}(\varphi(t, x))$ and

$$x_{I(t_j)}(t) < \min_{i \in O(t_j)} x_i(t) = \max_{i \in O(t_j)} x_i(t) = \bar{x}_\infty.$$

A similar proof holds for $\underline{\mathcal{I}}(\varphi(t, x))$, just by replacing the direction of inequalities and min with the max operator. The next result refines the claims of Lemma 4.

***Lemma** 5:* Assume $\bar{x}_\infty > \underline{x}_\infty$. Let there be an interval $J \subseteq \mathbb{R}$ with non-empty interior and $x \in \omega(x_0)$, such that $\bar{\mathcal{I}}(\varphi(t, x))$ and $\underline{\mathcal{I}}(\varphi(t, x))$ are both constant for $t \in J$. Then $\bar{\mathcal{I}}(\varphi(t, x))^c$ and $\underline{\mathcal{I}}(\varphi(t, x))^c$ are traps of the associated Petri Net (for $t \in J$) (denote them by T_1 and T_2) such that $T_1 \cup T_2 = P$ and the following inclusions hold:

$$\begin{array}{rcl} O(T_1 \backslash T_2) \cap I(T_2 \backslash T_1) & \subseteq & I(T_1 \backslash T_2) \\ O(T_2 \backslash T_1) \cap I(T_1 \backslash T_2) & \subseteq & I(T_2 \backslash T_1). \end{array} \quad (7)$$

Proof. By definition, provided $\bar{x}_\infty > \underline{x}_\infty$, we have $\underline{\mathcal{I}}(\varphi(t,x)) \subseteq \bar{\mathcal{I}}(\varphi(t,x))^c$; hence it follows $T_1 \cup T_2 = \bar{\mathcal{I}}(\varphi(t,x))^c \cup \underline{\mathcal{I}}(\varphi(t,x))^c \supseteq \bar{\mathcal{I}}(\varphi(t,x)) \cup \bar{\mathcal{I}}(\varphi(t,x))^c = P$. Moreover, both T_1 and T_2 are traps, by virtue of Lemma 4. We only need to prove the mutual inclusions in (7). Assume by contradiction that some $t_j \in O(T_1 \backslash T_2) \cap I(T_2 \backslash T_1)$ exists, such that $t_j \notin I(T_1 \backslash T_2)$. Then $O(t_j) \subseteq T_2$. In particular then, $\min_{i \in O(t_j)} x_i(t) \geq \min_{i \in T_2} x_i(t) > \underline{x}_\infty = x_{I(t_j)}(t)$. For any agent i such that $i \in O(t_j)$ we see that:

$$\dot{x}_i(t) \leq h_i(0, 0, \ldots, g_{I(t_j) \to O(t_j)}(x(t)), \ldots, 0) < 0,$$

which contradicts the fact that agents achieving the maximum have constant value for all t. A symmetric argument holds for the second inclusion in (7).

We are now ready to prove Theorem 1.

A. *Proof of main result*

We argue by contradiction. Assume that for some initial condition x_0 the solution of (5) does not achieve asymptotic consensus. By Lemma 2 we know that nevertheless solutions are uniformly bounded, so that the set $\omega(x_0)$ is non-empty and the limits $\bar{x}_\infty$ and $\underline{x}_\infty$ exist finite. Pick any $x \in \omega(x_0)$, and consider the associated solution $\varphi(t,x)$, along with the sets $\bar{\mathcal{I}}(\varphi(t,x))$ and $\underline{\mathcal{I}}(\varphi(t,x))$. By Lemma 3 the latter are monotonically non-increasing in time with respect to set inclusion (and non-empty), so that they will be constant after some finite time t (due to their finite cardinality). Hence, by Lemma 5, the sets $T_1 := \bar{\mathcal{I}}(\varphi(t,x))^c$ and $T_2 := \underline{\mathcal{I}}(\varphi(t,x))^c$ are found to be a pair of covering traps, which moreover fulfill inclusions (7). This however contradicts the Covering Traps Compatibility Property and completes the proof of the Theorem.

V. A NECESSARY CONDITION FOR CONSENSUS-ABILITY

We propose next an alternative topological condition which is necessary for asymptotic consensus to hold from arbitrary initial conditions $x_0 \in \mathbb{R}^N$. The necessary condition slightly relaxes the Covering Traps Compatibility condition by allowing an additional property to hold.

***Definition** 5:* (*Covering Traps Necessary Compatibility*) We say that a network fulfills the covering traps necessary compatibility if, for any pair of covering traps $T_1 \neq T_2$ (viz. $T_1 \cup T_2 = P$) at least one of the following conditions hold:

1) $O(T_2 \backslash T_1) \cap I(T_1 \backslash T_2) \nsubseteq I(T_2 \backslash T_1)$;
2) $O(T_1 \backslash T_2) \cap I(T_2 \backslash T_1) \nsubseteq I(T_1 \backslash T_2)$;
3) $O((T_1 \cap T_2)^c) \nsubseteq I((T_1 \cap T_2)^c)$.

The necessary compatibility condition relaxes the requirements on covering traps, by allowing conditions 1. and 2. to be violated, provided 3. is fulfilled. This condition allows to devise an arrangement of agents which are in equilibrium away from consensus as seen in the following proof.

***Theorem** 2:* Consider an arbitrary Petri Net, and the associated set of differential equations, as defined in equation (5). Then, a necessary condition for the solution of (5) to achieve asymptotic consensus irrespective of the initial condition $x_0 \in \mathbb{R}^N$ is that the the Covering Traps Necessary Compatibility holds for the associated Petri Net.

A. *Proof of Necessary condition*

We argue by contradiction. Assume that the Covering Traps Necessary Compatibility does not hold. We know that there exist covering traps $T_1 \neq T_2$ such that neither 1., 2. nor 3. in Definition 5 hold. Hence, we have all inclusions:

$$\begin{array}{rcl} O(T_1 \backslash T_2) \cap I(T_2 \backslash T_1) & \subseteq & I(T_1 \backslash T_2), \\ O(T_2 \backslash T_1) \cap I(T_1 \backslash T_2) & \subseteq & I(T_2 \backslash T_1), \\ O((T_1 \cap T_2)^c) & \subseteq & I((T_1 \cap T_2)^c). \end{array} \tag{8}$$

Consider the following initial condition $x = [x_1, \ldots x_N]'$ where x_i is defined below:

$$x_i = \begin{cases} 1 & \text{if } p_i \in T_1 \backslash T_2 \\ -1 & \text{if } p_i \in T_2 \backslash T_1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that agents in $T_1 \backslash T_2$ take the maximum value 1. Agents in $T_2 \backslash T_1$ take the minimum value -1 while agents in $T_1 \cap T_2$ take intermediate values. We claim that $\varphi(t,x) = x$ for all $t \geq 0$. This can be done by checking that every interaction $t_j \in T$ fulfills $g_{I(t_j) \to O(t_j)}(x) = 0$.
We consider several cases:

1) $t_j \in I(T_1 \backslash T_2)$. If, in addition, $t_j \in O(T_1 \backslash T_2)$ the conclusion is trivial, as all agents in $T_1 \backslash T_2$ share the maximum value 1, therefore $g_{I(t_j) \to O(t_j)}(x) = 0$. If, on the other hand, $t_j \in O(T_2 \backslash T_1)$ we have, $t_j \in O(T_2 \backslash T_1) \cap I(T_1 \backslash T_2) \subseteq I(T_2 \backslash T_1)$. Since all agents in $T_2 \backslash T_1$ share the same minimum value -1, this yields: $g_{I(t_j) \to O(t_j)}(x) = 0$. Finally, if $t_j \in O(T_1 \cap T_2)$ we know that $t_j \in O(T_2) \subseteq I(T_2)$ because T_2 is a trap, and therefore denoting $p_i \in T_1 \cap T_2$ such that $t_j \in O(p_i)$, we see that $x_i = 1/2$ is within the convex-hull of values of agents influenced by t_j. Therefore $g_{I(t_j) \to O(t_j)}(x) = 0$.
2) $t_j \in I(T_2 \backslash T_1)$. This case is treated by a symmetric argument to the previous one.
3) $t_j \in I(T_1 \cap T_2)$. Assume, without loss of generality, $t_j \notin I(T_2 \backslash T_1) \cup I(T_1 \backslash T_2)$ as otherwise we can go back to one of the previous two cases. Notice that $I(T_2 \backslash T_1) \cup I(T_1 \backslash T_2) = I((T_2 \backslash T_1) \cup I(T_1 \backslash T_2)) = I((T_1 \cap T_2)^c)$. Hence, by the third inclusion in (8), we see $t_j \notin O((T_1 \cap T_2)^c)$. Equivalently, $t_j \in O(T_1 \cap T_2)$. Since all agents in $T_1 \cap T_2$ share the same state value (i.e. 0), we have $g_{I(t_j) \to O(t_j)}(x) = 0$.

This completes the proof of the claim. In particular, then, asymptotic consensus is not achieved and necessity of the condition is shown.

B. *Example: necessary condition not sufficient*

We consider the network with 5 agents and the following set of 9 1-to-2 interactions. Accordingly we associate the following lines of interactions:

$$1 \to \{2,3\}, 1 \to \{2,4\}, 1 \to \{3,4\},$$

$$5 \to \{2,3\}, 5 \to \{2,4\}, 5 \to \{3,4\},$$

$$2 \to \{1,5\}, 3 \to \{1,5\}, 4 \to \{1,5\}.$$

The traps in this network are given by $\{p_1, p_2, p_3\}$, $\{p_1, p_2, p_4\}$, $\{p_1, p_3, p_4\}$, $\{p_5, p_2, p_3\}$, $\{p_5, p_2, p_4\}$,

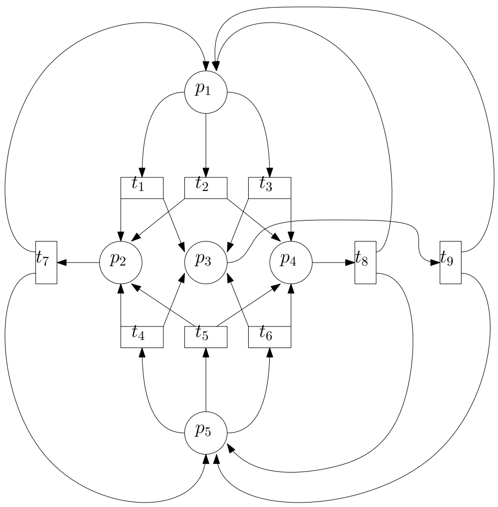

Fig. 5. A network fulfilling the necessary condition which does not achieve consensus

$\{p_5, p_3, p_4\}$, $\{p_1, p_2, p_3, p_4\}$, $\{p_2, p_3, p_4, p_5\}$ and P. It can be seen that traps $T_1 := \{p_1, p_2, p_3, p_4\}$, $T_2 := \{p_2, p_3, p_4, p_5\}$ violate the sufficient trap compatibility property. In fact $O(p_1) \cap I(p_5) = \emptyset$. On the other hand: $O(\{p_1, p_5\}) = \{t_1, t_2, t_3, t_4, t_5, t_6\} \not\subseteq \{t_7, t_8, t_9\} = I(\{p_1, p_5\})$. Hence the necessary trap compatibility condition is fulfilled. The network is shown in Fig. 5. It can be seen that the equations of this network are such that:

$$\dot{x}_1(t) - \dot{x}_5(t) = 0,$$

along any solution $x(t)$. Hence $x_1(t) - x_5(t) = x_1(0) - x_5(0)$ which violates asymptotic consensus provided $x_1(0) \neq x_5(0)$.

C. Discussion: linear time-varying embedding

Traditional consensus results have been estabilished for linear time-varying protocols, [3]. Such results are general enough to encompass the proposed setting. The goal of this discussion is to highlight the fundamental drawbacks that the analysis using standard graph-theoretical approaches and time-varying embedding is faced with. In particular, pursuing linear time-varying embedding in order to study consensus for networks involving 1-to-n interactions, one is faced with the challenging problem of interplay of state and time-dependence of links. Let $\mathcal{P}(N)$ denote the set of permutations of the first N integers. For each permutation of the N agents, $p_1, p_2, \ldots, p_N \in \mathcal{P}(N)$, one may consider the associated subset of state space:

$$X(p_1, p_2, \ldots, p_N) = \{x \in \mathbb{R}^N : x_{p_1} \geq x_{p_2} \geq \ldots \geq x_{p_N}\}.$$

These regions clearly cover the whole state-space, viz. $\bigcup_{p_1, p_2, \ldots, p_N \in \mathcal{P}(N)} X(p_1, \ldots, p_N) = \mathbb{R}^N$, moreover in each one of them we may determine, on the basis of the set of 1-to-n interactions, which ones will be "active", viz. give rise to some non-zero strength of influence between the influencing agent and the group of influenced agents. For instance, in the case of 3 agents, in the region $\{x : x_1 \leq x_2 \leq x_3\}$ the interaction $\{1\} \mapsto \{2, 3\}$ would be active, while the interaction $2 \mapsto \{1, 3\}$ would be disabled (since agent 2 is in the convex-hull of agents 1 and 3). In general, for each permutation in $P(N)$ one might define the active edges $E(p_1, \ldots, p_N)$ as follows:

$$\begin{aligned} E(p_1, p_2, \ldots p_N) &:= \{(p_i, p_j) \in P^2 : \exists t \in T \\ &: p_i \in I(t), i < j, \forall p_j \in O(t) \text{ or } i > j \, \forall p_j \in O(t)\}. \end{aligned}$$

One might conjecture that existence of a spanning tree in each of the graphs $G(p_1, \ldots, p_N) = (P, E(p_1, \ldots, p_N))$ would imply asymptotic consensus according to the results on linear time-varying systems, [3]. However such conditions are in a way too conservative as they do not account for the fact that it would be enough to have a spanning tree on the union of graphs encountered over a uniform shifting time-window along each solution $x(t)$. This kind of relaxation is hardly practical, however, due to the combinatorial explosion of permutations (as a function of N) which need to be considered, and the difficulties of a priori assessing the sequence (and duration of time intervals) in which the different $X(p_1, \ldots, p_N)$ regions are visited with the goal of building the associated union of graphs.

To further complicate matters, it turns out that even existence of a spanning tree in each of the graphs associated to the different regions $X(p_1, \ldots, p_n)$ is not enough to guarantee asymptotic consensus, due to the strict nature of inequalities needed in order to ensure a positive strength of influence in networks with 1-to-n interactions, as seen from the following example:

$$1 \mapsto \{2, 3\}, \quad 3 \mapsto \{1, 2\}. \tag{9}$$

This network is unable to achieve consensus from all initial conditions. For instance, starting with $x_1(0) = x_3(0) = 0$, $x_2(0) = 1$, yields the constant solution $x(t) = [0, 1, 0]'$. To better grasp the difference between 1-to-1 interactions and 1-to-n interactions consider the simplest possible set of equations corresponding to network (9):

$$\begin{array}{rcl} \dot{x}_1 &=& -\max\{\min\{x_1, x_2\} - x_3, 0\} \\ && +\max\{x_3 - \max\{x_1, x_2\}, 0\} \\ \dot{x}_2 &=& -\max\{\min\{x_1, x_2\} - x_3, 0\} \\ && +\max\{x_3 - \max\{x_1, x_2\}, 0\} \\ && -\max\{\min\{x_3, x_2\} - x_1, 0\} \\ && +\max\{x_1 - \max\{x_3, x_2\}, 0\} \\ \dot{x}_3 &=& -\max\{\min\{x_3, x_2\} - x_1, 0\} \\ && +\max\{x_1 - \max\{x_3, x_2\}, 0\} \end{array} \tag{10}$$

For equation (10), the set $\mathcal{X} = \{x : x_2 - x_3 \geq 1, x_1 \leq x_3 \leq 0\}$ can be proved to be forward invariant. In fact, this set belongs to $\{x : x_1 \leq x_3 \leq x_2\}$ where only the interaction $1 \mapsto \{2, 3\}$ can be active. Therefore, for all $x \in \mathcal{X}$, $\dot{x}_1 = 0$, and $\dot{x}_2 = \dot{x}_3 = -(x_3 - x_1)$. This allows to verify that Nagumo's forward invariance criterion is fulfilled.

Notice that this set does not contain consensus equilibria, as $x_2 - x_3 \geq 1$. Moreover it has positive and in fact infinite measure in $\mathbb{R}^3$, thus proving that lack of consensus appears for large segments of state-space. Explicit integration of the solutions is possible within $\mathcal{X}$, leading to: $x_2(t) - x_3(t) = x_2(0) - x_3(0)$, $x_1(t) = x_1(0)$ and $x_3(t) = x_1(0) + e^{-t}(x_3(0) - x_1(0))$, showing that only agent 1 and 3 achieve asymptotic consensus.

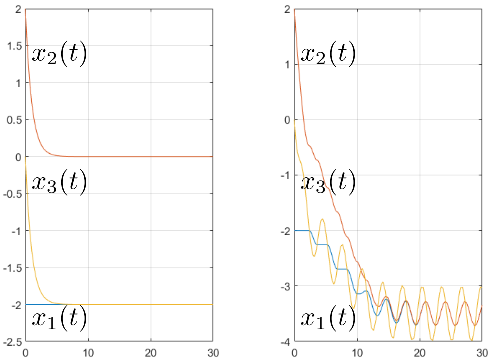

Fig. 6. Simulation of (10) without and with additive noise in x_3

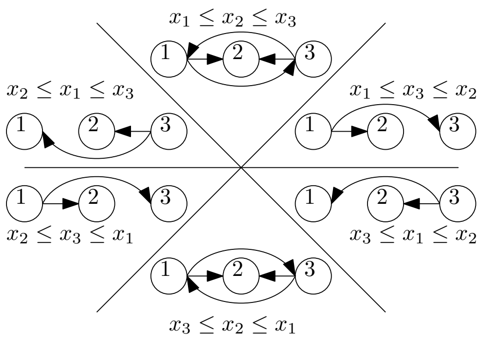

Fig. 7. Graphs associated to linear time-varying embedding

See Fig. 6, where a numerical simulation is shown for $x(0) = [-2, 2, 0]'$. While n-to-1 joint-agent interactions can be used to enforce resilient consensus in the face of malicious agents, [17], [18], the properties afforded by 1-to-n interactions in the presence of exogenous disturbances are more subtle. Somewhat surprisingly, at first sight, the presence of exogenous noise may favour consensus of agents non affected by the disturbance. We show below simulative evidence of this fact, by running the same simulation of (10) when agent 3 is affected by noise (in this case an additive sinusoidal disturbance). Examining the different regions $X(p_1, p_2, p_3)$, however, (see Fig. 7) shows that a spanning tree exists in every graph considered. The actual connectivity and strength of interaction is however overestimated by such graphs for values of x where two or more state variables are equal to each other, thus inducing a wrong conclusion by blindly applying Moreau's results (these in fact only work when a strictly positive lower-bound is available on the agents' coupling coefficient of active interactions).

VI. Conclusions and Future Work

The paper considered a novel scenario of networks with 1-to-n joint-agent interactions and formulated sufficient and necessary conditions to assess consensus-ability. Such interactions could represent and extend the heterophilous behaviour among networks' agents in social and opinion networks. Future research directions include the investigation of the filtering property of 1-to-n interactions-based protocols and their applicability to engineering context, the extension to networks with m-to-n joint interactions as well as the weakening of consensus-ability to p-order clustering conditions. Notice, in particular, that extension to the case of n-to-m interactions appears to be extremely challenging. Merging definitions based on siphons (n-to-1 case) and definitions based on traps (1-to-n case) is a completely open issue even just at the graphical level. This is needed to formalize a notion encompassing both the so called Siphon Overlapping property and Traps Compatibility property. Finally, connecting this property with the dynamics of the resulting differential equations appears to involve significant technical challenges.

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